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Operator formalism on general algebraic curves

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Abstract

The usual Laurent expansion of the analytic tensors on the complex plane is generalized to any closed and orientable Riemann surface represented as an affine algebraic curve. As an application, the operator formalism for the $b - c$ systems is developed. The physical states are expressed by means of creation and annihilation operators as in the complex plane and the correlation functions are evaluated starting from simple normal ordering rules. The Hilbert space of the theory exhibits an interesting internal structure, being splitted into n (n is the number of branches of the curve) independent Hilbert spaces. In this way we are able to realize new kinds of conformal field theories at genus zero with symmetry group $Vir^n \otimes G$, Vir being the Virasoro group and G denoting a discrete and nonabelian crystallographic group. Exploiting the operator formalism a large collection of explicit formulas of string theory is derived. Finally, we develop as an important byproduct new methods in order to handle differential equations related to monodromy, like the Riemann monodromy problem.

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1. Introduction

In this paper the Laurent expansion of analytic tensors defined on the complex plane is generalized to any algebraic curve associated to a Weierstrass polynomial of the kind $F(z, y) = 0$ with $z \in \mathbf{CP}_1$. The elements of the basis in which the tensors are expanded are explicitly represented in terms of the multivalued function $y(z)$ and of the Weierstrass

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polynomial itself. As an application, the fermionic $b - c$ systems [1] with integer spin $\lambda \geq 2$ are quantized expanding the fields in generalized Laurent series and requiring that the coefficients of the series are annihilation and creation operators. This establishes an operator formalism for the $b - c$ systems on general algebraic curves and eventually, due to the conformal equivalence between Riemann surfaces and algebraic curves [2], on any Riemann surface. It is thus remarkable that we are able to derive here all the correlation functions of the $b-c$ systems starting from simple commutation relations between operators on the complex plane. Another surprising feature of the $b - c$ systems, which we will show in this work, is that the physical space of the $b - c$ theory splits into a finite number of independent Hilbert spaces. For each Hilbert space it is possible to define a separate vacuum and the creation and annihilation operators acting on different Hilbert spaces do commute. In this way we are able to answer with explicit examples the questions (see, e.g. [3] and references therein) about the structure of the Hilbert space in free conformal field theories defined on Riemann surfaces. The splitting of the Hilbert space is also crucial for proving that there are no spurious singularities in the physical amplitudes of the $b - c$ systems. This was the main obstacle in extending the approach of Refs. [4,5] to general algebraic curves of genus greater than four like those treated in this work.

The physical applications of our operator formalism are in string theory, conformal field theories and integrable models. On one side, we provide explicit formulas in string theory, to be compared with the results obtained from other explicit methods, e.g. [6–8]. On the other side, our choice of working on algebraic curves establishes a link between string theory and conformal field theories and integrable models on the plane, as will be explained with greater details below and in the Conclusions. In particular, we are able to realize in this paper the $b - c$ systems on Riemann surfaces in terms of a new kind of conformal field theories, with symmetry group $Vir^n \otimes G$, where Vir is the Virasoro group and G is a discrete crystallographic group provided by the monodromy group of the algebraic curve. This is an important step towards a nonperturbative treatment of string theory, as conjectured in Refs. [9,10]. Finally, free conformal field theories on algebraic curves can be also considered as toy models in the study of field theories in curved space–times. Analogies with quantum field theories in the presence of wormholes and the appearance of pseudo-particles with nonstandard statistics has been explained in Ref. [11].

As already remarked, one of the most important results obtained from our operator formalism is that we provide explicit formulas for the ghost amplitudes in bosonic string theory. Starting from the original paper of [1], these amplitudes have been computed and generalized on any Riemann surface using various techniques. A partial list of these works is contained in Refs. [6–8] and [12–16]. Our approach is different in the sense that it is based on the representation of Riemann surfaces as n-sheeted coverings of the complex plane. With respect to other operator formalisms, this choice has many advantages in the case of the $b - c$ systems. First of all, the bases introduced here in order to expand the fields are multivalued bases on the complex sphere, which become single valued only on the algebraic curve. In this way, the relationship between the usual bases exploited at genus zero and those used at higher genera is immediate. Moreover, all the correlation functions of the $b - c$ systems are obtained exploiting simple normal ordering rules on the complex sphere. Their expression is not given as an infinite sum over the elements of the basis, but coincides exactly with the form yielded by the fermionic construction of Refs. [8], explicitly taking care of the zero modes also when $\lambda \geq 2$. Another bonus of our method is its explicitness. The λ differentials involved in the correlation functions are simply written as rational functions in terms of the variables z and $y(z)$ which parametrize the algebraic curve, so that it is not necessary to express them in terms of theta functions or automorphic forms. It is important to stress at this point that the representation of Riemann surfaces as algebraic curves provides interesting perturbative results in string theory, see for example Refs. [17] in the case of hyperelliptic curves. Nevertheless, we would also like to emphasize the strong connections established by our work between conformal field theories on the complex plane and the $b - c$ systems on Riemann surfaces. The investigations in this direction started in Refs. [9,10] in the case of algebraic curves with Z_n symmetry. In Refs. [18,19] and [20] these results have been extended to the algebraic curves with nonabelian group of symmetry D_n, showing that an interesting braid group statistics arises in the amplitudes of the $b - c$ systems. In this paper we are able to realize for the first time the $b - c$ systems in terms of a new class of conformal field theories with symmetry $Vir^n \otimes G$ for any nonabelian monodromy group G.

Despite the fact that here we are mainly focused in the construction of the Hilbert space of the $b - c$ systems, there are many other possible applications of our operator formalism, for instance in those research areas where monodromy properties play a fundamental role, like in the theory of integrable models and conformal field theories. Concerning integrable models, the basis given here for the $b - c$ fields generalizes an analogous one introduced in [21] in order to explain why the solutions of the level zero $sl(2)$ Knizhnik–Zamolodchikov equations [22] are given in terms of hyperelliptic integrals. With our basis it seems possible to extend the results of Ref. [21] also to the level zero $sl(n)$ Knizhnik–Zamolodchikov equations, which are satisfied by the form factors of the $su(n)$-invariant Thirring model and whose solutions are related to the periods of differentials on algebraic curves with Z_n group of symmetry [23]. The explicit formulas given in this paper can also shed some light on the conjectured connections between general algebraic curves and integrable models [23–25]. As a matter of fact, the results established here, like for instance the methods developed in Appendix A and Proposition 1, provide very efficient techniques to treat multivalued functions and differential equations related to monodromy.

This paper is organized as follows. In Section 2 the usual Laurent expansions are extended to general algebraic curves of genus g. In order to point out that there is a superposition of independent modes in the physical amplitudes of the $b - c$ systems, which leads to the splitting of the Hilbert space, two different Laurent expansions are introduced for the b and c fields separately. The completeness and equivalence of both bases in the space of the analytic tensors defined on Σ_g is shown in Proposition 1. Furthermore, the elements of the two bases are explicitly derived in terms of the coordinates z and $y(z)$ and their divisors are computed as well. For technical simplicity, for example in the computation of the genus and of the relevant divisors, we consider nondegenerate algebraic curves, but the outcome will turn out to be completely general. In Section 3 we introduce the operator formalism for the $b-c$ systems with $\lambda > 1$. The case of $\lambda = 1$ is complicated by the presence of the zero mode in the field c and will be treated in a separate paper. We verify the splitting of the Hilbert space in a finite number of independent sectors. The normal ordering between the fields is a simple generalization of the usual normal ordering on the complex plane. All the correlation functions of the $b-c$ systems on an algebraic curve are then obtained as vacuum expectation values of multivalued fields. In Section 4 we check the operator formalism computing the n-point functions of the $b-c$ systems on a quartic of genus three and on the Z_n symmetric curves. These curves represent, together with other few examples, the only cases in which the results are already known by exploiting other methods [4,5,9,10]. Finally, in Section 5 we present the conclusions and discuss the possible future developments.

2. Generalized Laurent expansions on algebraic curves

We consider here the (class of) algebraic curves Σ_g associated to the vanishing of the following Weierstrass polynomial:

$$F(z, y) = P_n(z)y^n + P_{n-1}(z)y^{n-1} + \cdots + P_1(z)y + P_0(z) = 0, \tag{2.1}$$

where the $P_j(z) = \sum_{m=0}^{n-j} \alpha_{j,m} z^m$ are polynomials in z of degree at most $n-j$, $j = 0, \ldots, n$ and the $\alpha_{j,m}$ are complex parameters. y can be viewed as a multivalued function on the complex sphere $\mathbb{C}\mathrm{P}_1$ or as a meromorphic function on the Riemann surface defined by Eq. (2.1). The function $y(z)$ has poles only at $z = \infty$. By suitably setting to zero some of the parameters $\alpha_{j,m}$ it is possible to obtain every other kind of Weierstrass polynomials. However, in order to fix the ideas, though it will not be strictly necessary in the following, we suppose that the algebraic curves treated here are nondegenerate. We recall that, in order to define nondegenerated curves, one has to introduce a homogeneous polynomial $\tilde{F}(z, y, w)$ of degree n in the three variables z, y and w defined in such a way that $\tilde{F}(z, y, 1) = F(z, y)$. In our case it is sufficient to change the polynomials $P_j(z)$'s into $\tilde{P}_j(z, w) = \sum_{m=0}^{n-j} \alpha_{j,m} z^m w^{n-j-m}$. The nondegeneracy condition amounts to the requirement that the equations:

$$\tilde{F}(z, y, w) = \tilde{F}_y(z, y, w) = \tilde{F}_z(z, y, w) = \tilde{F}_w(z, y, w) = 0, \tag{2.2}$$

where $\tilde{F}_y(z, y, w) \equiv \partial_w \tilde{F}(z, y, w)$, $\tilde{F}_z(z, y, w) \equiv \partial_z \tilde{F}(z, y, w)$ and $\tilde{F}_w(z, y, w) \equiv \partial_w \tilde{F}(z, y, w)$ are never simultaneously satisfied.

Since Eq. (2.1) is an overdetermined system of algebraic equations, it can be satisfied only for very particular choices of the parameters $\alpha_{j,m}$. Therefore, the nondegenerate solutions of Eq. (2.1) still describe a huge class of algebraic curves. Remarkably, the genus of the equivalent Riemann surface can be directly computed from the form of the Weierstrass polynomial (2.1). This is Baker's method, explained in Ref. [26, Vol. I, p.404]. It turns out that the genus of Σ_g is $g = \frac{1}{2}(n-1)(n-2)$.

On the algebraic curves (2.1) we consider a theory of free fermionic $b-c$ systems with integer spin $\lambda \geq 2$ defined by the following action:

$$S_{bc} = \int_{\Sigma_g} \mathrm{d}^2\xi \left(b\bar{\partial}c + \text{c.c.}\right), \tag{2.3}$$

where ξ and $\bar{\xi}$ are complex coordinates on Σ_g. An introduction of the effective operatorial formalism should follow from the suitable identification of the classical degrees of freedom of the fields b and c. For that purpose let us analyze the solutions of the classical equations of motion descending from Eq. (2.3):

$$\bar{\partial}b = \bar{\partial}c = 0. \tag{2.4}$$

We shall expand them in the following basis:

$$b(z)\,\mathrm{d}z^{\lambda} = \sum_{k=0}^{n-1}\sum_{i=-\infty}^{\infty} b_{k,i} z^{-i-\lambda} f_k(z)\,\mathrm{d}z^{\lambda}, \tag{2.5}$$

$$c(z)\,\mathrm{d}z^{1-\lambda} = \sum_{k=0}^{n-1}\sum_{i=-\infty}^{\infty} c_{k,i} z^{-i+\lambda-1} \phi_k(z)\,\mathrm{d}z^{1-\lambda} \tag{2.6}$$

with f_k and ϕ_l chosen as follows $(k, l = 0, \ldots, n-1)$:

$$f_k(z) = \frac{y^{n-1-k}(z)\,\mathrm{d}z^{\lambda}}{(F_y(z, y(z)))^{\lambda}}, \tag{2.7}$$

$$\phi_l(w)\,\mathrm{d}w^{1-\lambda} = \frac{\mathrm{d}w^{1-\lambda}}{(F_y(w, y(w)))^{1-\lambda}} \Big(y^l(w) + y^{l-1}(w)P_{n-1}(w) + y^{l-2}(w)P_{n-2}(w) + \cdots + P_{n-l}(w)\Big). \tag{2.8}$$

We notice that we have introduced two different expansions for the fields b and c. This is only for the sake of convenience in the formulation of the operatorial formalism. In fact, after the replacement $\lambda \to 1-\lambda$, the basis

$$B_{i,k}(z, y(z))\,\mathrm{d}z^{\lambda} = z^i f_k(z)\,\mathrm{d}z^{\lambda}, \quad i = 0, 1, \ldots, \quad k = 0, \ldots, n-1 \tag{2.9}$$

used in Eq. (2.5) for the λ-differentials turns out to be equivalent to the basis

$$C_{i,k}(w, y(w))\,\mathrm{d}w^{1-\lambda} = z^i \phi_k(w)\,\mathrm{d}w^{1-\lambda}, \quad i = 0, 1, \ldots, \quad k = 0, \ldots, n-1 \tag{2.10}$$

used in the case of the $1-\lambda$ differentials. In other words, the elements $B_{i,k}(z, y(z))\,\mathrm{d}z^{\mu}$ are linear combinations of the $C_{i,k}(z, y(z))\,\mathrm{d}z^{1-\lambda}$ and vice versa if $\mu = 1-\lambda$ as it is easy to prove. The reasons of taking asymmetric expansions for the fields b and c will be clear when the correlation functions are computed, showing that they are multilinear superpositions of the elements $B_{i,k}(z, y(z))\,\mathrm{d}z^{\lambda}$ and $C_{i,k}(z, y(z))\,\mathrm{d}z^{1-\lambda}$. For the moment, we just state the following proposition, which will be proved in Appendix A.

Proposition 1. *Every λ-differential $\omega\,\mathrm{d}z^{\lambda}$, $\lambda \in \mathbb{Z}$, on an arbitrary algebraic curve Σ_g can be expanded in terms of the basis whose elements are given by Eq.* (2.9) *as follows*:

$$\omega(z)\,\mathrm{d}z^{\lambda} = \sum_{k=0}^{n-1} g_k(z) f_k(z, y(z))\,\mathrm{d}z^{\lambda}, \tag{2.11}$$

where $f_k(z, y(z))$ has been defined in Eq. (2.7) *and the $g_k(z)$'s are single valued, rational functions of z. An analogous statement is also true for the basis* (2.10).

Hence, Eqs. (2.5) and (2.6) can be regarded as the generalization on algebraic curves of the standard Laurent expansion on the complex plane. This is a great advantage with respect to the usual Poiseaux expansions, defined in the neighborhood of a point and, consequently, only locally valid. If the $P_j(z)$ are set to zero for $j = 1, \ldots, n-1$, the limiting case of Z_n symmetric algebraic curves is obtained. It is then possible to check that the expansions (2.5) and (2.6) become equal to those obtained in Ref. [11].

In order to ascertain that the physical amplitudes do not contain spurious poles and to compute the zero modes, it is necessary to determine the analytical properties of the elements of the bases (2.9) and (2.10). The best strategy is to derive first the divisors of their building blocks, i.e. of $\mathrm{d}z$, y and $F_y(z, y)$. These divisors can be easily evaluated on any algebraic curve using the methods of Ref. [4], but unfortunately there is no universal procedure which applies automatically to any kind of Weierstrass polynomials, in particular to those corresponding to degenerate curves. The branch points of the curve Σ_g are determined by the conditions:

$$F(z, y) = F_y(z, y) = 0. \tag{2.12}$$

Eliminating y from the two equations written above one finds an equation in the variable z of the kind $r(z) = 0$. $r(z)$ is the resultant of the system (2.12) and it is a polynomial expressed in terms of the $P_j(z)$, $j = 0, \ldots, n$, appearing in Eq. (2.1). This can be seen using for instance the dialitic method of Sylvester (see Ref. [2, Vol. II, p.79]). Performing the calculations explicitly it turns out that, in general, the degree of $r(z)$ is equal to $n(n-1)$. In this case there are no branch points at infinity as we shall suppose throughout this paper for the sake of simplicity. This implies that there are n_{bp} finite branch points $a_1, \ldots, a_{n_{bp}}$ of multiplicity ν_i, where ν_i describes the number of branches of y connected at the branch point a_i. The multiplicities and the number of branch points satisfy the following Riemann–Hurwitz equation:

$$2g - 2 = -2n + \sum_{i=1}^{n_{bp}} (\nu_i - 1). \tag{2.13}$$

The algebraic curves in which all the branch points have multiplicity two are said to be in the normal form and are well known in the mathematical literature [26,2]. For example, the corresponding Riemann surfaces can be explicitly constructed in terms of branch points and branch lines and even a basis of independent homology cycles is known (Lüroth Theorem). It is clear that surfaces in the normal form are particular cases of our general scheme.

At this point, denoting the branch points by a_i, $i = 1, \ldots, n_{bp}$ and following the methods explained in Ref. [4], we obtain the desired divisors:

$$[\mathrm{d}z] = \sum_{p=1}^{n_{bp}} (\nu_p - 1)a_p \; - \; 2\sum_{j=0}^{n-1} \infty_j, \tag{2.14}$$

$$[y] = \sum_{r=1}^{n} q_r \; - \; \sum_{j=0}^{n-1} \infty_j, \tag{2.15}$$

$$[F_y] = \sum_{p=1}^{n_{bp}} (\nu_p - 1)a_p \; - \; (n-1)\sum_{j=0}^{n-1} \infty_j. \tag{2.16}$$

In Eq. (2.15) the q_j denote the zeros of y which, when projected from the Riemann surface on the z complex plane, coincide with the zeros of $P_0(z)$. Moreover, ∞_j describes the projection of the point at infinity on the jth sheet of the Riemann surface. Finally, in our conventions positive and negative integers denote the order of the zeros and of the poles respectively. Starting from Eqs. (2.14)–(2.16) it is possible to find the divisors of the elements of the basis (2.9):

$$[z^i f_k] = (n-1-k)\sum_{s=1}^{n} q_s + i\sum_{l=0}^{n-1} 0_l + (k + \lambda(n-3) - (n-1) - i)\sum_{l=0}^{n-1} \infty_l, \tag{2.17}$$

where, using the notation exploited for the point at infinity, 0_l denotes the projection of the point $z = 0$ on the jth sheet. An analogous formula can be derived for the $C_{i,k}(w, y(w))\,\mathrm{d}z^{1-\lambda}$.

To conclude this section we explicitly derive the form of the zero modes associated to the equations of motion (2.4). We try the following ansatz:

$$\Omega_{k,i}\,\mathrm{d}z^{\lambda} = f_k(z)z^{-i-\lambda}. \tag{2.18}$$

After the substitution $i \to -i - \lambda$ in Eq. (2.17), we see that $\Omega_{k,i}\,\mathrm{d}z^{\lambda}$ has no singularities whenever

$$i \leq -\lambda \tag{2.19}$$

and

$$k + \lambda(n-2) - (n-1) + i \geq 0. \tag{2.20}$$

In order to find all the possible zero modes we proceed by fixing the value of k in Eq. (2.20) and then computing the possible values of i compatible also with Eq. (2.19). Skipping the simple cases in which $n = 2, 3$ corresponding to genus zero and one Riemann surfaces, we obtain the following results:

(1) $n > 4$, $\lambda > 1$ or $n = 4$, $\lambda > 2$. Then the $N_b = (2\lambda - 1)(g - 1)$ independent zero modes are of the form given by Eq. (2.18) and

$$k = 0, \ldots, n-1, \qquad \lambda(2-n) + n - 1 - k \leq i \leq -\lambda \tag{2.21}$$

Moreover, the number of zero modes proportional to f_k is given by

$$N_{b_k} = \lambda(n-3) + k - n + 2. \tag{2.22}$$

(2) $n = 4$, $\lambda = 2$. In this case the genus of the curve is three and the six independent zero modes occur when

$$k = 1, 2, 3, \qquad -1 - k \leq i \leq -2 \tag{2.23}$$

and

$$N_{b_k} = k, \quad k = 1, 2, 3. \tag{2.24}$$

(3) $n > 3$, $\lambda = 1$. Here the conditions determining all the g independent zero modes of the form (2.18) are

$$k = 2, 3, \ldots, n-1, \qquad -k + 1 \leq i \leq -1, \tag{2.25}$$

while

$$N_{b_k} = k - 1, \quad k = 2, \ldots, n-1. \tag{2.26}$$

(4) Finally, if $\lambda = 0$, we have only one zero mode provided by the constant function in the c fields. Using the basis (2.10) this zero mode is obtained for $k = i = 0$.

In the first two cases written above it is easy to check that total number of zero modes is

$$N_b = \sum_{k=0}^{n-1} N_{b_k} = \frac{(2\lambda - 1)(n-3)n}{2}, \tag{2.27}$$

which in fact expresses the well known Riemann–Roch theorem for a surface of genus g.

3. The operator formalism

In Section 2 we have shown how to expand the classical $b-c$ fields on an algebraic curve in generalized Laurent series. Now we quantize the fields transforming the coefficients $b_{i,k}$ and $c_{i,k}$ of Eqs. (2.5) and (2.6) into creation and annihilation operators. In the following, it will be convenient to exploit the basis (2.9) in order to expand the b fields for $\lambda > 1$ and the basis (2.10) in order to expand the c fields. This is just a matter of convenience since the bases (2.9) and (2.10) have no particular distinguishing properties. In principle, one could also expand the $b-c$ fields in terms of another basis, but this would require a certain amount of complications since the zero modes and all the other tensors entering the theory are most naturally expressed as linear combinations of the elements (2.9) and (2.10). At this point we divide the degrees of freedom of the $b-c$ systems into n sectors, numbered by the index $k = 0, \ldots, n-1$ and characterized by the tensors f_k for the b fields and by the tensors ϕ_k for the c fields. This seems a natural choice, since all the zero modes are expressed in a simple way in terms of f_k. In fact, we have seen in Section 2 that it is even possible to define the numbers N_{b_k} of the zero modes corresponding to $f_k(z)$. Further, we assume that in our operator formalism the modes $z^j f_k(z)$ labeled by different indices k do not interact. This hypothesis, to be proven a posteriori, implies that the space on which the $b-c$ fields defined on a Riemann surface act can be decomposed into a set of n independent Hilbert spaces if the Riemann surface is represented as an n-sheeted branch covering over $\mathbb{CP}_1$. Accordingly, we quantize the theory (2.3) postulating the following basic anticommutation relations for the coefficients $b_{k,i}$ and $c_{k,i}$ appearing in Eqs. (2.5) and (2.6) respectively:

$$\{b_{k,i}, c_{k',i'}\} = \delta_{kk'}\delta_{i+i',0}. \tag{3.1}$$

For each value of $k = 0, \ldots, n-1$ these creation and annihilation operators act on the vacuum $|0\rangle_k$, $k = 0, \ldots, n-1$, which represents the usual vacuum of the $b-c$ systems on the complex plane. The "total vacuum" of the $b-c$ systems on Σ_g is given by

$$|0\rangle = \bigotimes_{k=0}^{n-1} |0\rangle_k. \tag{3.2}$$

We demand that

$$b^-_{k,i}|0\rangle_k \equiv b_{k,i}|0\rangle_k = 0, \quad k = 0, \ldots, n-1, \; i \geq 1-\lambda, \tag{3.3}$$

$$c^-_{k,i}|0\rangle_k \equiv c_{k,i}|0\rangle_k = 0, \quad k = 0, \ldots, n-1, \; i \geq \lambda, \tag{3.4}$$

Moreover, we introduce the "out" vacua ${}_k\langle 0|$ requiring that

$${}_k\langle 0|b^+_{k,i} \equiv {}_k\langle 0|b_{k,i} = 0, \quad k = 0, \ldots, n-1, \; i \leq -\lambda - N_{b_k}, \tag{3.5}$$

$${}_k\langle 0|c^+_{k,i} \equiv {}_k\langle 0|c_{k,i} = 0, \quad k = 0, \ldots, n-1, \; i \leq \lambda - 1. \tag{3.6}$$

From the above equations we see that some operators $b_{k,j}$ correspond to zero modes and the remaining ones are organized in two sets of creation and annihilation ones. The annihilation operators annihilate states with negative energy as it is possible to verify from Eq. (2.5). The same applies to the $c_{k,j}$ with the only difference that there are no zero modes for them.

Finally, let us introduce the following useful notations

$$b_k(z)\,\mathrm{d}z^\lambda = f_k(z) \sum_{i=-\infty}^{\infty} b_{k,i} z^{-i-\lambda}\,\mathrm{d}z^\lambda, \tag{3.7}$$

$$c_k(z)\,\mathrm{d}z^{1-\lambda} = \phi_k(z) \sum_{i=-\infty}^{\infty} c_{k,i} z^{-i+\lambda-1}\,\mathrm{d}z^{1-\lambda}, \tag{3.8}$$

$$b(z)\,\mathrm{d}z^\lambda = \sum_{k=0}^{n-1} b_k(z)\,\mathrm{d}z^\lambda, \qquad c(z)\,\mathrm{d}z^{1-\lambda} = \sum_{k=0}^{n-1} c_k(z)\,\mathrm{d}z^{1-\lambda}. \tag{3.9}$$

From the above considerations, by exploiting the commutation relations (3.1) and the natural definition of the "normal ordering" of the $b-c$ systems, we get

$$b_k(z)c_k(w) = {:}b_k(z)c_k(w){:} + \frac{1}{z-w} f_k(z)\phi_k(w). \tag{3.10}$$

The "time ordering" is implemented by the requirement that the fields $b(z)$ and $c(w)$ are radially ordered with respect to the variables z and w. Finally, in order to take into account the zero modes, we impose the following conditions:

$$ {}_k\langle 0|0\rangle_k = 0 \quad \text{if } N_{b_k} \neq 0; \qquad {}_k\langle 0|\prod_{i=1}^{N_{b_k}} b_{k,i}|0\rangle_k = 1. \tag{3.11} $$

Starting from Eqs. (3.1)–(3.11) it is now possible to compute on any algebraic curve the correlation functions of the $b - c$ systems as expectation values over the vacuum (3.2) of the fields b and c defined in Eqs. (3.9). However the numbers N_{b_k} of the zero modes must be individually computed for each different class of algebraic curves.

As a first step, we calculate the following amplitudes:

$$ {}_k\langle 0|b_k(z_1)\dots b_k(z_{N_{b_k}})|0\rangle_k = \mathrm{Det}\left|\Omega_{k,j}(z_i)\right|, \quad i,j = 1,\dots,N_{b_k} \tag{3.12} $$

$$ \langle 0|b(z_1)\dots b(z_{N_b})|0\rangle = \det\left|\Omega_I(z_J)\right|, \tag{3.13} $$

where the vacuum $|0\rangle$ and the fields $b(z)\,\mathrm{d}z^\lambda$ have already been defined in Eqs. (3.2) and (3.9). Moreover $I, J = 1, \dots, \sum_{k=0}^{n-1} = N_b$, N_b denoting the total number of zero modes. Finally the $\Omega_I(z)\,\mathrm{d}z^\lambda$ represents all the possible zero modes with spin λ.

$$ \Omega_I(z)\,\mathrm{d}z^\lambda \in \left\{\Omega_{k,i}(z)\,\mathrm{d}z^\lambda \mid 1 \le i \le N_{b_k}, 0 \le k \le n-1\right\}. $$

The zero modes $\Omega_{k,j}(z)\,\mathrm{d}z^\lambda$ can be obtained in terms of z and y from Eqs. (2.18) and (2.7):

$$ \Omega_{k,j}(z) \equiv \frac{y^{n-1-k} z^{j-1}}{(F_y(z, y(z)))^\lambda}\,\mathrm{d}z^\lambda \tag{3.14} $$

where the range of the indices $k = 0, \dots, n-1$ and $j = 1, \dots, N_{b_k}$ is given in Eqs. (2.22) and (2.24).

The proof of Eq. (3.12) is straightforward, while Eq. (3.13) will be proven in Appendix B. We notice that Eq. (3.13), obtained here in a pure operatorial way from the basic commutation relations (3.1) satisfied by the creation and annihilation operators $b_{k,i}$ and $b_{k,i}$, is in complete agreement with the standard results [8].

Now we are ready to compute the propagator of the $b - c$ fields, which, in our formalism, is defined by the following ratio of correlators:

$$ G_\lambda(z, w) = \frac{\langle 0|b(z)c(w)\prod_{I=1}^{N_b} b(z_I)|0\rangle}{\langle 0|\prod_{I=1}^{N_b} b(z_I)|0\rangle}. \tag{3.15} $$

From Eq. (3.10) the normal ordering of any two fields b and c becomes

$$ b(z)c(w) = :b(z)c(w): + K_\lambda(z, w)\,\mathrm{d}z^\lambda\,\mathrm{d}w^{1-\lambda}, \tag{3.16} $$

where $K_\lambda(z, w)$ denotes the following tensor:

$$ (z - w)K_\lambda(z, w) = \sum_{j=0}^{n-1} f_j(z)\phi_j(w) \tag{3.17} $$

In order to fix the overall sign in the expression of the propagator we suppose that the fields contained in the correlators appearing in Eq. (3.15) are already radially ordered, i.e.

$$|z| > |w| > |z_1| \cdots > |z_{N_b}|.$$

Thus, applying Eq. (3.16), we obtain

$$\begin{aligned}
&\frac{\langle 0|b^{(l)}(z)c^{(l')}(w)\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle}{\langle 0|\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle} \\
&= K_\lambda^{(ll')}(z,w)\,\mathrm{d}z^\lambda\,\mathrm{d}w^{1-\lambda} + \left(\sum_{J=1}^{N_b}(-1)^J K_\lambda^{(l_J l')}(z_J,w)\right. \\
&\quad\left.\times\frac{\langle 0|b^{(l_1)}(z_1)\ldots b^{(l_{J-1})}(z_{J-1})b^{(l)}(z)b^{(l_{J+1})}(z_{J+1})\ldots b^{(l_{N_b})}(z_{N_b})|0\rangle}{\langle 0|b^{(l_1)}(z_1)\ldots b^{(l_{N_b})}(z_{N_b})|0\rangle}\right),
\end{aligned} \tag{3.18}$$

where l, l' and l_I, l_J denote the branches of fields and tensors in the variables z, w, z_I, z_J respectively. The residual correlation functions in Eq. (3.18) contain products of N_b fields b and therefore can be easily computed by means of Eq. (3.13). The final result is the following propagator:

$$\frac{\langle 0|b^{(l)}(z)c^{(l')}(w)\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle}{\langle 0|\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle}
= \frac{\det\begin{vmatrix} \Omega_1^{(l)}(z) & \ldots & \Omega_{N_b}^{(l)}(z) & K_\lambda^{(ll')}(z,w) \\ \Omega_1^{(l_1)}(z_1) & \ldots & \Omega_{N_b}^{(l_1)}(z_1) & K_\lambda^{(l_1 l')}(z_1,w) \\ \vdots & \ddots & \vdots & \vdots \\ \Omega_1^{(l_{N_b})}(z_{N_b}) & \ldots & \Omega_{N_b}^{(l_{N_b})}(z_{N_b}) & K_\lambda^{(l_{N_b} l')}(z_{N_b},w) \end{vmatrix}}{\det|\Omega_I(z_J)|}. \tag{3.19}$$

Eq. (3.19) is the final expression of the propagator of the $b-c$ systems evaluated on a general algebraic curve. Of course, we have still to check that the tensor appearing in the right-hand side of Eq. (3.19) has only a physical simple pole at the points $z = w$ and $w = z_J$, $J = 1, \ldots, N_b$. Moreover, since the propagator is a multivalued tensor, these singularities must occur when the branches coincide, i.e. when $l = l'$ and $l' = l_J$, respectively. To verify that the propagator (3.18) has the correct singularities, we apply the divisors (2.15)–(2.17) to the tensor $K_\lambda(z,w)$, which is responsible for the divergences. In this way we restrict ourselves to the general curves of genus $g = \frac{1}{2}(n-1)(n-2)$. However, it is possible to verify the absence of spurious poles also for the other curves. Some examples will be reported in Section 4.

First of all we treat the case (1) explained in Section 2, i.e. when Eq. (2.21) is valid. In this case, apart from the factor $1/(z-w)$, $K_\lambda(z,w)$ is a linear combination of the λ-differentials $f_k(z)$, which are all zero modes. From Eq. (2.17) it is in fact easy to see that the singularities of $K_\lambda(z,w)$ in the variable z can occur only when $z = w$. In the variable

w, instead, in addition to the singularities in $w = z$ there are spurious poles at infinity given by the $1 - \lambda$ differentials $\phi_k(w)$. Of course these spurious poles must not contribute in the final expression of the propagator. To show this, we rewrite $K_\lambda(z, w)$ as follows:

$$K_\lambda(z, w) = \sum_{k=0}^{n-1} u_k(z, w),$$

where

$$u_k(z, w) = \frac{1}{z - w}\phi_k(w) f_k(z).$$

We obtain the following behavior for $u_k(z, w)$ when $w \to \infty$:

$$u_k(z, w) \sim w^{\lambda(n-3)+k-n+2} + \cdots \tag{3.20}$$

Therefore, there is a danger of spurious poles whenever $\lambda(n - 3) + k - n + 2 > 0$. Let us call $u_k(z, w)_{\text{div}}$ the divergent terms in the asymptotic expansion at $w = \infty$ of $u_k(z, w)$. From (3.20), in order to compute $u_k(z, w)_{\text{div}}$, we need to expand the quantities $1/(z - w)$ and $y(w)$ up to the order $w^{-(1+\lambda(n-3)+k-n+1)}$ and substitute them back in the expression of $u_k(z, w)$.

$$\frac{1}{z - w} \sim \frac{1}{w}\left(1 + \frac{z}{w} + \cdots + \left(\frac{z}{w}\right)^{\lambda(n-3)+k-n+1}\right) + \cdots \tag{3.21}$$

Moreover, since from Eq. (2.15) $y(w)$ is not branched in $w = \infty$, we have

$$y(w) \sim \sum_{s=-(\lambda(n-3)+k-n+2)}^{1} \gamma_s w^s + \cdots, \tag{3.22}$$

where the coefficients γ_i, $i = 1, \ldots, -(\lambda(n - 3) + k - n + 2)$ can be computed in terms of the parameters $\alpha_{j,m}$ by direct insertion of the right-hand side of the above equation in Eq. (2.1). Using the expansions (3.21) and (3.22) we obtain that $u_k(z, w)_{\text{div}}$ is of the following form:

$$\begin{aligned} u_k(z, w)_{\text{div}} = \Big(&\beta_{\lambda(n-3)+k-n+2} w^{\lambda(n-3)+k-n+2} + \beta_{\lambda(n-3)+k-n+1} w^{\lambda(n-3)+k-n+1} z \\ &+ \cdots + \beta_1 w z^{\lambda(n-3)+k-n+1}\Big) f_k(z) \end{aligned} \tag{3.23}$$

Eq. (3.23) shows that the divergent part of $u_k(z, w)$ is proportional to a λ-differential in z of the kind $\Omega_{i,k}\, \mathrm{d}z^\lambda = z^i f_k(z)$, $i = 0, \ldots, \lambda(n - 3) + k - n + 1$, $k = 0, \ldots, n - 1$. On the other side, after performing the substitution $-i - \lambda \to i$ in Eq. (2.17), it is possible to see that the $\Omega_{i,k}\, \mathrm{d}z^\lambda$ are zero modes if the indices i and k are taken as above. As an upshot, looking at Eq. (3.19), which is a determinant of a matrix whose columns contain all the independent zero modes, it turns out that the contribution given by the terms carrying the spurious poles at $w = \infty$ in $K_\lambda(z, w)$ is zero:

$$\frac{\det\begin{vmatrix} \Omega_1^{(l)}(z) & \dots & \Omega_{N_b}^{(l)}(z) & u_k^{(ll')}(z,w)_{\mathrm{div}} \\ \Omega_1^{(l_1)}(z_1) & \dots & \Omega_{N_b}^{(l_1)}(z_1) & u_k^{(l_1 l')}(z_1,w)_{\mathrm{div}} \\ \vdots & \ddots & \vdots & \vdots \\ \Omega_1^{(l_{N_b})}(z_{N_b}) & \dots & \Omega_{N_b}^{(l_{N_b})}(z_{N_b}) & u_k^{(l_{N_b} l')}(z_{N_b},w)_{\mathrm{div}} \end{vmatrix}}{\det|\Omega_I(z_J)|} = 0.$$

This concludes our proof that there are no singularities at $w = \infty$ in the propagator (3.19). However, the proof is valid only if Eq. (2.21) is true. The exception is provided by the 2-differentials on an algebraic curve of genus three (see Eq. (2.23)). Taking $n = 4$ and $\lambda = 2$ in Eq. (2.1), in fact, we have for $k \geq 1$:

$$u_k(z,w)_{\mathrm{div}} = \left(\beta_k w^k + \cdots + \beta_1 w z^{k-1}\right) f_k(z).$$

These terms are again proportional to zero modes in z by Eq. (2.23) and, exploiting the previous arguments, the spurious poles at $w = \infty$ do not appear in Eq. (3.19). When $k = 0$, instead, both tensors $\phi(w)\ f_0(z)$ are linearly divergent at infinity. Fortunately, thanks to the term $1/(z - w)$, $u_0(z, w)$ does not show up spurious poles in z and w.

Finally, let us check that the singularity in $z = w$ of $K_\lambda(z, w)$ occurs only when $l = l'$ and it is a simple pole. The proof will be done without assuming any restriction to the form of the Weierstrass polynomial (2.1). Let us start substituting in the expression (3.17) of $K_\lambda(z, w)$ the explicit form of f_k and ϕ_k given by the definitions (2.7) and (2.8):

$$\begin{aligned} K_\lambda(z,w) = {} & \frac{[F_y(w,y(w))]^{\lambda-1}}{[F_y(z,y(z))]^\lambda}\frac{1}{z-w}\mathrm{d}z^\lambda \mathrm{d}w^{1-\lambda} \\ & \times\Big[y^{n-1}(z) + y^{n-2}(z)\,(y(w) + P_{n-1}(w)) \\ & + y^{n-3}(y^2(w) + P_{n-1}(w)y(w) + P_{n-2}(w)) \\ & + \cdots + y^{n-1}(w) + y^{n-2}(w)P_{n-1}(w) + \cdots + P_1(w)\Big], \end{aligned} \tag{3.24}$$

where it is understood that $y(z)$ and $y(w)$ are in the branch l and l', respectively. It is now possible to write

$$\begin{aligned} & y^{n-1}(z) + y^{n-2}(z)(y(w) + P_{n-1}(w)) + y^{n-3}(y^2(w) + P_{n-1}(w)y(w) + P_{n-2}(w)) \\ & + \cdots + y^{n-1}(w) + y^{n-2}(w)P_{n-1}(w) + \cdots + P(w) = \frac{F(w,y(z))}{y(z)-y(w)}. \end{aligned} \tag{3.25}$$

Eq. (3.25) has been obtained multiplying and dividing the left-hand side by $y(z) - y(w)$ and then exploiting the formulas $F(w, y(w)) = 0$ and

$$a^n - b^n = (a-b)(b^{n-1} + ab^{n-2} + a^2b^{n-3} + \cdots + a^{n-1}).$$

With these simplifications it turns out that

$$K_\lambda^{(ll')}(z,w) = \frac{[F_y(w,y^{l'}(w))]^{\lambda-1}}{[F_y(z,y^l(z))]^\lambda}\frac{1}{z-w}\frac{F(w,y^l(z))}{y^l(z)-y^{l'}(w)}\,\mathrm{d}z^\lambda\,\mathrm{d}w^{1-\lambda}. \tag{3.26}$$

But this is the well known form of the Weierstrass kernel on algebraic curves (see for instance Ref. [4]) which has exactly one pole in $z = w$ for $l = l'$ and spurious poles in $w = \infty$. Since the latter spurious poles are irrelevant in the final expression of the propagator as previously shown, we conclude that Eq. (3.19) yields the correct two point function of the $b-c$ systems with $\lambda > 1$ on the algebraic curves of the kind (2.1). Eq. (3.26) shows that the Weierstrass polynomial is a superposition of the multivalued modes $f_k(z)$ and $\phi_k(w)$ and it is a crucial result for the operator formalism. As a matter of fact, once the Weierstrass polynomial and the zero modes are given, it is possible to construct the correlation of the $b-c$ systems automatically using the formulas given below. It is thus remarkable that both the zero modes and the Weierstrass kernel exhibit the splitting into n sectors characterized by the tensors f_k and ϕ_k, $k = 0, \ldots, n-1$.

Starting from Eq. (3.19), we are able to compute all the other n-point functions applying the Wick theorem. The Wick theorem for the $b-c$ systems has been rigorously studied in [15] and it is valid also in our case. One can check it inductively. As previously shown, the Wick theorem based on the normal ordering (3.16) holds in the case of the two point function (3.19). Now let us suppose that the Wick theorem has been verified for the correlator

$$
\begin{aligned}
&G_{N-1,M-1}(z_1, \ldots, z_{N-1}; w_1, \ldots, w_{M-1}) \\
&\quad = \langle 0|b(z_1)\ldots b(z_{N-1})c(w_1)\ldots c(w_{M-1})|0\rangle
\end{aligned}
$$

with $N - M = N_b$. Then, using Eq. (3.16) we obtain for $G_{N,M}(z_1, \ldots, z_N; w_1, \ldots, w_M)$:

$$
\begin{aligned}
&\langle 0|b(z_N)c(w_M)b(z_1)\ldots b(z_{N-1})c(w_1)\ldots c(w_{M-1})|0\rangle \\
&\quad = \sum_{i=1}^{M}(-1)^i K_\lambda(z_N, w_i) \\
&\qquad \times G_{N-1,M-1}(z_1, \ldots, z_{N-1}; w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{M-1}).
\end{aligned}
\tag{3.27}
$$

All the other possible contractions vanish due to the fact that the Wick theorem holds by hypothesis in the case of any product containing $N-1$ fields b and $M-1$ fields c. As an upshot we obtain

$$
\begin{aligned}
&\left\langle \prod_{s=1}^{M} b^{\,(l_s)}(z_\rho) \prod_{t=1}^{N} c^{(l'_t)}(w_t) \right\rangle \\
&\quad = \det \begin{vmatrix}
\Omega_1^{(l_1)}(z_1) & \ldots & \Omega_{N_b}^{(l_1)}(z_1) & K_\lambda^{(l_1 l'_1)}(z_1, w_1) & \ldots & K_\lambda^{(l_1 l'_N)}(z_1, w_N) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Omega_1^{(l_M)}(z_M) & \ldots & \Omega_{N_b}^{(l_M)}(z_M) & K_\lambda^{(l_M l'_1)}(z_M, w_1) & \ldots & K_\lambda^{(l_M l'_N)}(z_M, w_N)
\end{vmatrix},
\end{aligned}
\tag{3.28}
$$

where $M - N = (2\lambda - 1)(g - 1) = N_b$. The tensor $K_\lambda^{(ll')}(z, w)$ has spurious poles in the limit $w \to \infty$. However one can show as before that these poles do not contribute to the determinant (3.28). The important fact to be noted here is that both Eqs. (3.19) and (3.28) were evaluated using a pure operator formalism on the complex plane. This operator formalism, however, reproduces exactly the correlation functions of the $b-c$ systems on general algebraic curves. Exploiting the correspondence between algebraic curves and Riemann surfaces, we can also say that Eqs. (3.19) and (3.28) represent the explicit form of the formulas given in Refs. [8].

4. The operator formalism on a general quartic of genus three and on the Z_n symmetric curves

In this section we treat two relatively simple cases of algebraic curves for which the results can be compared with those obtained by applying other methods [5,4,9,10,27]. The first example is the quartic Q_3 of genus three with Weierstrass polynomial:

$$y^3 + 3p(z)y - 2q(z) = 0, \tag{4.1}$$

where $p(z)$ and $q(z)$ are polynomials of degree three and four respectively. This curve represents all the possible Riemann surfaces of genus three that are nonhyperellitpic. $y(z)$ has nine finite branch points $a_1, \ldots, a_9$, which are roots of the equation $p^3(z) + q^2(z) = 0$. Moreover, there is one branch point at infinity of multiplicity two. By analysing this example we demonstrate that the operator formalism applies also to curves with branch points at infinity and only for technical reasons we considered in the general discussion the more restricted case in which all the branch points are finite. As a convention, we suppose that the 0th and 1st sheets are joined at infinity, so that $\infty_0 = \infty_1 = \infty_{01}$. The relevant divisors are [4]:

$$[\mathrm{d}z] = \sum_{p=1}^{9} a_p - 3\infty_{01} - 2\infty_2, \tag{4.2}$$

$$[y] = \sum_{j=1}^{4} q_j - 3\infty_{01} - \infty_2, \tag{4.3}$$

$$[F_y] = \sum_{p=1}^{9} a_p - 6\infty_{01} - 3\infty_2, \tag{4.4}$$

where $F_y(z, y(z)) = 3(y(z) + p(z))$. Now we notice that the Weierstrass polynomial (4.1) can be obtained from Eq. (2.1) after setting

$$P_4(z) = P_2(z) = 0, \qquad P_3(z) = 1, \qquad P_1(z) = p(z), \qquad P_0(z) = -2q(z).$$

Since the generalized Laurent expansions (2.5) and (2.6) are valid for any algebraic curve, it is sufficient to perform the above substitution in Eqs. (2.5)–(2.8), with $k = 0, 1, 2$. As a consequence, the multivalued modes f_k and ϕ_k become on Q_3:

$$f_k(z) = \frac{y^{2-k}(z)\,\mathrm{d}z^\lambda}{[F_y(z, y(z))]^\lambda}, \quad k = 0, 1, 2, \tag{4.5}$$

$$\phi_k(w) = \frac{y^k(w)\,\mathrm{d}w^{1-\lambda}}{[F_y(w, y(w))]^{1-\lambda}}, \quad k = 0, 1, \tag{4.6}$$

$$\phi_2(w) = \frac{(y^2(w) + 3p(w))\,\mathrm{d}w^{1-\lambda}}{[F_y(w, y(w))]^{1-\lambda}}. \tag{4.7}$$

At this point we start to work out the operator formalism for $\lambda = 2$. In this way the notations are simpler and it is possible to better illustrate the way in which the operator formalism works, carrying out all the calculations in details. Apart from the evaluation of the relevant divisors, the particular case under consideration is just a subcase of the general operator formalism of Section 3 with $n = 3$ and $\lambda = 2$, so that the space of the $b - c$ fields on Q_3 is splitted into three independent Hilbert spaces. Therefore, we can exploit without modifications Eqs. (3.1)–(3.11) in order to compute the correlation functions of the $b - c$ systems. The only difference from section 3 is in the number of zero modes, for which we have to use the divisors (4.2)–(4.4). After a straightforward calculation, we get the explicit expression of the six independent quadratic differentials on Q_3:

$$\Omega_{0,0} = f_0(z),$$

$$\Omega_{1,j}(z)\,\mathrm{d}z^2 = z^{j-1} f_1(z), \quad j = 1, 2,$$

$$\Omega_{2,j}\,\mathrm{d}z^2 = z^{j-1} f_2(z), \quad j = 1, 2, 3.$$

As a consequence the number of zero modes in the different k-sectors becomes

$$N_{b_0} = 1, \qquad N_{b_1} = 2, \qquad N_{b_2} = 3. \tag{4.8}$$

Now we are ready to prove Eq. (3.13). To this purpose, we start from the simplest nonvanishing correlator (3.13), which contains in this case six zero modes:

$$\langle 0|b(z_1)\dots b(z_6)|0\rangle \equiv \langle 0| \prod_{i=1}^{6} \left(\sum_{k=0}^{2} b_k(z_i) \right) |0\rangle. \tag{4.9}$$

The product in the right-hand side of (4.9) can be expanded in all possible monomials of the fields b_k using Lemma B.1. Due to the condition (3.11), only the monomials containing three fields b_2, two fields b_1 and one field b_0 survive after normal ordering. It is straightforward to check that these monomials are given by the following formula:

$$\begin{aligned}\langle 0|b(z_1)\dots b(z_6)|0\rangle = \sum_\sigma \operatorname{sign}(\sigma)\ {}_2\langle 0| \prod_{l_2=1}^{3} b_2(z_{\sigma(l_2)})|0\rangle_2 \\ \times {}_1\langle 0| \prod_{l_1=4}^{5} b_1(z_{\sigma(l_1)})|0\rangle_1\, {}_0\langle 0|b_0(z_{\sigma(6)})|0\rangle_0,\end{aligned} \tag{4.10}$$

where the sum over σ runs over all permutation of the integers $1, \dots, 6$, ordered in such a way that

$$\sigma(1) < \sigma(2) < \sigma(3), \qquad \sigma(4) < \sigma(5).$$

The vacuum expectation values remaining in Eq. (4.10) can be directly computed from the definitions (2.5) and (3.11) or using Eq. (3.12). In either cases the result is

$$\langle 0|b(z_1)\dots b(z_6)|0\rangle = \sum_\sigma \mathrm{sign}(\sigma)\det\left|\Omega_{2,j_2}(z_{\sigma(l_2)})\right| \det\left|\Omega_{1,j_1}(z_{\sigma(l_1)})\right| \Omega_{0,1}(z_{\sigma(6)})$$

with $j_2, l_2 = 1,2,3$ and $j_1, l_1 = 4,5$. It is now easy to recognize that the right-hand side of the above equation is the determinant of the six holomorphic quadratic differentials, taken at the points $z_1, \dots, z_6$ and expanded as shown in Lemma B.2. This concludes the proof of Eq. (3.13) on the curve Q_3:

$$\langle 0|b(z_1)\dots b(z_6)|0\rangle = \det|\Omega_I(z_J)| \tag{4.11}$$

Finally, we compute the two point function. The creation and annihilation operators are defined as in Eqs. (3.3)–(3.6) after the substitutions $\lambda = 2$ and $n = 3$. Moreover, the number of zero modes in the different k-sectors is given by Eq. (4.8). As a consequence, the normal ordering (3.16) becomes

$$b(z)c(w) = :b(z)c_k(w): + K_2(z,w)\,\mathrm{d}z^2\,\mathrm{d}w^{-1}. \tag{4.12}$$

From the definition (3.17) of $K_2(z,w)$ we have

$$\begin{aligned} &K_2(z,w)\,\mathrm{d}z^2\,\mathrm{d}w^{-1} \\ &\quad= \frac{\mathrm{d}z^2\,\mathrm{d}w^{-1}}{z-w}\left[\frac{F_y(w,y(w))}{F_y(z,y(z))}\right]\left(\frac{y^2(z)+y(z)y(w)+y^2(w)+3p(w)}{F_y(z,y(z))}\right). \end{aligned}$$

The final form of $K_2(z,w)\,\mathrm{d}z^2\,\mathrm{d}w^{-1}$ is obtained multiplying and dividing the right-hand side of the above equation by $y(z)-y(w)$:

$$K_2(z,w)\,\mathrm{d}z^2\,\mathrm{d}w^{-1} = \frac{\mathrm{d}z^2\,\mathrm{d}w^{-1}}{z-w}\left[\frac{F_y(w,y(w))}{F_y(z,y(z))}\right]\frac{F(w,y(z))}{(y(z)-y(w))F_y(z,y(z))}, \tag{4.13}$$

where $F(z,y(z))$ denotes the Weierstrass polynomial (4.1). The proof that $K_2^{(ll')}(z,w)\,\mathrm{d}z^2\,\mathrm{d}w^{-1}$ is a good Weierstrass kernel with only one simple pole in $z = w$ and $l = l'$ was already given in Ref. [4] and we do not report it here. Since there are no spurious poles, we can compute the two point function of the $b-c$ system with $\lambda = 2$ using the Wick theorem as shown in Eq. (3.18):

$$\begin{aligned} &\frac{\langle 0|b^{(l)}(z)c^{(l')}(w)\prod_{I=1}^6 b^{(l_I)}(z_I)|0\rangle}{\langle 0|\prod_{I=1}^6 b^{(l_I)}(z_I)|0\rangle} \\ &\quad= \frac{\det\begin{vmatrix} \Omega_1^{(l)}(z) & \dots & \Omega_6^{(l)}(z) & K_2^{(ll')}(z,w) \\ \Omega_1^{(l_1)}(z_1) & \dots & \Omega_3^{(l_1)}(z_1) & K_2^{(l_1l')}(z_1,w) \\ \vdots & \ddots & \vdots & \vdots \\ \Omega_1^{(l_3)}(z_3) & \dots & \Omega_3^{(l_3)}(z_3) & K_2^{(l_3l')}(z_3,w) \end{vmatrix}}{\det|\Omega_I(z_J)|} \end{aligned} \tag{4.14}$$

with $I, J = 1, \ldots, 6$. This is exactly the explicit propagator of the ghost of string theory derived in Ref. [4]. In order to derive also the higher-order correlation functions it is just sufficient to exploit the Wick theorem as explained in Section 3.

Now we pass to the Z_n curves of the kind:

$$y^n = \prod_{i=1}^{nm}(z - a_i), \tag{4.15}$$

where n and m are integers. Strictly speaking, this class of curves is degenerate in the sense of Eq. (2.2). It is thus interesting to verify through this example that the operator formalism applies also to more general curves than those treated in Section 2.

The Z_n curves can be obtained from Eq. (2.1) after the following substitutions: $P_n(z) = 1$, $P_j(z) = 0$, $j = 1, \ldots, n-1$ and $P_0(z) = \prod_{i=1}^{nm}(z - a_i)$. The points $a_i \in \mathbb{C}$ are the branch points of the curve, so that $n_{bp} = nm$. The genus of the Z_n curves (4.15) is given by

$$g = 1 - n + \tfrac{1}{2}nm(n-1) \tag{4.16}$$

and the relevant divisors are:

$$[\mathrm{d}z] = (n-1)\sum_{p=1}^{nm} a_p - 2\sum_{j=0}^{n-1}\infty_j,$$

$$[y] = \sum_{p=1}^{nm} a_p - m\sum_{j=0}^{n-1}\infty_j,$$

$$[F_y] = (n-1)\sum_{p=1}^{nm} a_p - (n-1)m\sum_{j=0}^{n-1}\infty_j.$$

For simplicity, we consider here only Riemann surfaces of genus $g \geq 2$, i.e. neither the torus nor the sphere, for which an operator formalism is already known. The generalized Laurent expansions (2.5) and (2.6) become in this case:

$$b(z)\,\mathrm{d}z^\lambda = \sum_{k=0}^{n-1}\sum_{i=-\infty}^{\infty} b_{k,i} z^{-i-\lambda} f_k(z)\,\mathrm{d}z^\lambda, \tag{4.17}$$

$$c(z)\,\mathrm{d}z^{1-\lambda} = \sum_{k=0}^{n-1}\sum_{i=-\infty}^{\infty} c_{k,i} z^{-i+\lambda-1}\phi_k(z)\,\mathrm{d}z^{1-\lambda}, \tag{4.18}$$

where

$$f_k(z)\,\mathrm{d}z^\lambda = \frac{\mathrm{d}z^\lambda}{[y(z)]^{-k+\lambda(n-1)}}, \quad k = 0, \ldots, n-1, \tag{4.19}$$

$$\phi_k(z)\,\mathrm{d}z^{1-\lambda} = \frac{\mathrm{d}z^{1-\lambda}}{[y(z)]^{k-\lambda(n-1)}}, \quad k = 0, \ldots, n-1. \tag{4.20}$$

The last ingredient of the operator formalism is provided by the zero modes, which are of the form

$$\Omega_{k,j}\,\mathrm{d}z^{\lambda} = \frac{z^{j-1}\,\mathrm{d}z^{\lambda}}{[y(z)]^{-k+\lambda(n-1)}}, \quad j = 1, \ldots, N_{b_k}, \tag{4.21}$$

where $N_{b_k} = -2\lambda+1+\lambda(n-1)m-km$. It is easy to check that $\sum_{k=0}^{n-1} N_{b_k} = (2\lambda-1)(g-1)$, giving exactly the number of the zero modes predicted by the Riemann–Roch theorem. A treatment of the $b-c$ systems on the Z_n curves can now be performed using the techniques explained in Section 3. The creation and annihilation operators are defined by the following relations:

$$b^-_{k,i}|0\rangle_k \equiv b_{k,i}|0\rangle_k = 0, \quad k = 0, \ldots, n-1, \; i \geq 1-\lambda, \tag{4.22}$$

$$c^-_{k,i}|0\rangle_k \equiv c_{k,i}|0\rangle_k = 0, \quad k = 0, \ldots, n-1, \; i \geq \lambda, \tag{4.23}$$

$${}_k\langle 0|b^+_{k,i} \equiv {}_k\langle 0|b_{k,i} = 0, \quad k = 0, \ldots, n-1, \; i \leq -\lambda - N_{b_k}, \tag{4.24}$$

$${}_k\langle 0|c^+_{k,i} \equiv {}_k\langle 0|c_{k,i} = 0, \quad k = 0, \ldots, n-1, \; i \leq \lambda - 1. \tag{4.25}$$

Finally, the normal ordering of the product of two fields $b(z)c(w)$ is given by

$$b_k(z)c_k(w) = \,:b_k(z)c_k(w):+\frac{1}{z-w}f_k(z)\phi_k(w), \tag{4.26}$$

$$c_k(z)b_k(w) = \,:c_k(z)b_k(w):+\frac{1}{z-w}f_k(w)\phi_k(z). \tag{4.27}$$

so that the Weierstrass kernel takes the form

$$K_\lambda(z,w)\,\mathrm{d}z^{\lambda}\,\mathrm{d}w^{1-\lambda} = \frac{\mathrm{d}z^{\lambda}\,\mathrm{d}w^{1-\lambda}}{z-w}\sum_{k=0}^{n-1}\left[\frac{y(w)}{y(z)}\right]^{\lambda(n-1)-k} \tag{4.28}$$

Moreover, in order to take into account also the zero modes, we impose again the condition (3.11) on the vacua $|0\rangle_k$:

$${}_k\langle 0|0\rangle_k = 0 \quad \text{if } N_{b_k} \neq 0; \qquad {}_k\langle 0|\prod_{i=1}^{N_{b_k}} b_{k,i}|0\rangle_k = 1.$$

With the above definitions, we can easily prove Eqs. (3.12) and (3.13) using the formulas of Appendix B. Exploiting the normal ordering (4.26) and (4.27) we obtain the final formula of the propagator in a purely operatorial way:

$$\frac{\langle 0|b^{(l)}(z)c^{(l')}(w)\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle}{\langle 0|\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle}$$

$$= \frac{\det \begin{vmatrix} \Omega_1^{(l)}(z) & \dots & \Omega_{N_b}^{(l)}(z) & K_\lambda^{(ll')}(z,w) \\ \Omega_1^{(l_1)}(z_1) & \dots & \Omega_{N_b}^{(l_1)}(z_1) & K_\lambda^{(l_1 l')}(z_1,w) \\ \vdots & \ddots & \vdots & \vdots \\ \Omega_1^{(l_{N_b})}(z_{N_b}) & \dots & \Omega_{N_b}^{(l_{N_b})}(z_{N_b}) & K_\lambda^{(l_{N_b} l')}(z_{N_b},w) \end{vmatrix}}{\det|\Omega_I(z_J)|}, \tag{4.29}$$

where the Weierstrass kernel $K_\lambda(z,w)$ has been defined in Eq. (4.28). Also in the case of the Z_n curves, one can easily show that the spurious poles of $K_\lambda(z,w)$ at $w = \infty$ are harmless. Finally the higher-order correlation functions can be obtained as in the general case from the Wick theorem. Formally, they look like those in Eq. (3.28) apart from the fact that the Weierstrass kernel is that of Eq. (4.28) and the zero modes are those of Eq. (4.21). The results obtained with the operator formalism in the Z_n case can be compared with the analogous results obtained in a purely geometrical way in Ref. [27].

5. Conclusions

We have seen in this paper that the correlation functions of the $b-c$ systems with integer $\lambda > 1$ can be computed on any nondegenerate algebraic curve using a pure operator formalism. However, we would like to stress that our operator formalism applies to arbitrary algebraic curves. This claim has been verified for instance in the case of the Z_n algebraic curves (4.15) which are degenerate with respect to the definition (2.2) given in Section 2. Moreover, in Section 4 we have also considered the example of the quartic Q_3 with a branch point at infinity. The essential point of our construction is based on the fact that the relevant modes propagating in the amplitudes are multivalued fields defined on the complex plane. Crucial points of our approach are the generalized Laurent expansion explained in Section 2 and the splitting of the space of the $b-c$ fields into n independent Hilbert spaces. The splitting directly follows from the fact that both the zero modes and the Weierstrass kernel, which are the main ingredients in order to construct the correlation functions (see Eq. (3.28)), are superpositions of the multivalued modes $f_k(z)$ and $\phi_k(w)$ given in Eqs. (2.7) and (2.8). For the zero modes, this is a straightforward consequence of Proposition 1. For the Weierstrass kernel, instead, this has been proved in Eq. (3.26). In both cases, no assumption on the form of the Weierstrass polynomial was made. As an upshot, Eq. (3.28) provides the explicit form of the correlation functions of the $b-c$ systems on arbitrary closed and orientable Riemann surfaces. Once the Weierstrass polynomial is given, the Weierstrass kernel $K_\lambda(z,w)$ can be computed from Eq. (3.26) in terms of the coordinates z and y. The derivation of the zero modes, instead, is more complicated, since there are no universal procedures which yield the zero modes on any algebraic curve without previously knowing the divisors of the basic building blocks $\mathrm{d}z$, y and F_y. This is just a formal problem because, once the form of the Weierstrass polynomial is fixed, we are able to obtain the above divisors exploiting the techniques of Ref. [4]. However, due to the fact that it is not possible to cover in one paper all the existing families of curves, the explicit calculation of the zero modes and the proof that there are no spurious singularities in the amplitudes has been performed after imposing some assumptions on the Weierstrass polynomial. The chosen families of curves, however, are very general and do not exhibit any discrete group of symmetry.

Beyond the discussed results in string theory, we would like also to outline some interesting consequences following from our analysis in the study of conformal field theories and of the Riemann monodromy problem. To begin, we notice that the Laurent bases described in Section 2 represent a great advantage with respect to the usual Poiseaux series, which allow only local expansions of the multivalued analytic tensors. For instance, representing the variable z in polar coordinates, the bases (2.9) and (2.10) can be regarded as the global generalization of the usual Fourier series. The possibility of solving differential equations on any algebraic curve, also with boundary, by means of these series is currently under investigation. Moreover, starting from Eq. (2.7) and putting $\lambda = 0$, we are able to reproduce every multivalued function defined by a general Weierstrass polynomial. In this sense the Riemann monodromy problem [28], intended here as the problem of finding all the independent functions multivalued according to the monodromy data given by the branch points and the local monodromy group of an arbitrary algebraic curve, has been implicitly solved. For example, let us consider one of the simplest examples, the quartic Q_3 of genus three given in Eq. (4.1). In this case it is known that the Riemann monodromy problem admits three independent solutions whose zeros and poles must be concentrated at the branch points and at $z = \infty$ [28]. Two rationally independent functions fulfilling these constraints are $f_0(z) = 1$, the zero mode, and $f_2(z) = F_y(z, y(z))$ while, by Proposition 1, the third function should have the form given in Eq. (A.6) for $\lambda = 0$:

$$f_1(z) = g_0(z) + g_1(z)y(z) + g_2(z)y^2(z).$$

After a few calculations, one finds

$$f_1(z) = p(z)^2 + 2q(z)y - p(z)y^2.$$

As a matter of fact, $f_1(z)$ is proportional to the square of $f_2(z)$ which, as already remarked, has all its zeros at the branch points and the poles at infinity. Thus, once the form of the Weierstrass polynomial is known, the problem of constructing a complete set of independent functions with given zeros and poles at the branch points can be easily solved. The difficulties appear instead in reducing the differential equations satisfied by these functions to the standard form of the Riemann monodromy problem. [2]

The reason for which the differential equations satisfied by a rational function $f(z, y)$ of z and y become complicated is that $f(z, y)$ is defined on a Riemann surface. Therefore, the two-dimensional Poincaré group is explicitly broken. To explain this, we consider the equations of the Riemann monodromy problem as flux equations with complex time z:

[2] Conversely, in the usual methods of solving the Riemann monodromy problem, [28,29] the equations are in the standard form from the beginning, but the solutions are in the form of infinite series. When no exact convergence is guaranteed, the independent solutions are valid only locally and should be understood in terms of analytic continuation.

$$\frac{\mathrm{d}\Psi(z)}{\mathrm{d}z} = v(z)\Psi(z), \tag{5.1}$$

where $\Psi(z)$ is a vector of multivalued functions and $v(z) \equiv v_z(z)$ is a matrix of (possibly multivalued) vectors. We remember that the form of $v(z)$ is constrained by the requirement that all its singularities should be simple poles at the branch points and in $z = \infty$. It is now easy to see that on the complex plane, as well as on the Z_n symmetric curves, the only way of fulfilling this condition is given by

$$v_{ij}(z)\,\mathrm{d}z = \frac{A_{ij}\,\mathrm{d}z}{z - a_j}, \tag{5.2}$$

where the a_j represent the branch points and the A_{ij} are constants. Thus the vector $v(z)$ is invariant under the global translations of the coordinates z and a_j. On a general algebraic curve, however, Eq. (5.2) is not the only possibility. For example, using the divisors (2.14)–(2.16) we can immediately check that the following vector has only simple poles at the branch points

$$v_z(z) = \frac{\mathrm{d}z}{[F_y(z, y(z))]^2}. \tag{5.3}$$

Vectors of the kind (5.3) break explicitly the invariance under translations exhibited by Eq. (5.1) and lead in principle to nonlinear monodromy equations which seems to have not much in common with the familiar Fuchsian equations on the complex plane. Of course, these nonlinear equations must be reduced to the form (5.2), but this is not easy to achieve, since on a general algebraic curve there are no explicit functional relations between the branch points and the parameters appearing in the Weierstrass polynomial. Until now, the only class of algebraic curves with nonabelian group of symmetry for which the solutions of the Riemann monodromy problem have been explicitly constructed in terms of z and $y(z)$ are the D_n curves of Refs. [18–20]. The results obtained in [20] confirm that the derivatives with respect to z of these solutions contain vectors of the form (5.3). After some manipulations, however, the Riemann monodromy problem can be reduced to a set of equations given by Plemelj – see the first reference of [28] – which is equivalent to the usual Fuchsian formulation (5.2).

Concluding, the techniques explained in this work open the possibility of performing explicit calculations on any Riemann surface starting from simple normal ordering rules which generalize those of the $b-c$ systems on the complex plane. In this way, the correlation functions of the $b - c$ systems with $\lambda > 1$ have been computed without bosonization and the Hilbert space of the $b - c$ fields has been simply reproduced in terms of multivalued modes on the complex plane. It would be interesting to study with the same methods the Hilbert space of massless scalar fields. This case is however complicated by the fact that the theory is not completely conformal and there is a mixing between z and its complex conjugate variable $\bar{z}$ in the amplitudes. Another important line of investigation is to explore the connections of our formalism with that of Refs. [7], which also provides very explicit results using the Schottky parametrization of the Riemann surfaces. As a matter of fact, g-loop vertices as those defined in Refs. [7] could simulate the monodromy inside of the correlation functions of the $b-c$ systems, replacing the twist fields exploited in [11]. The applications in the Riemann monodromy problem discussed before are also relevant in the case of the Knizhnik–Zamolodchikov equations, which are of the same form (5.1). Also in the Knizhnik–Zamolodchikov equations, vectors of the kind (5.3) appear due to the explicit breaking of the two-dimensional Poincaré group on a Riemann surface. The way in which this affects the particle statistics inside the amplitudes has been studied on the D_n curves in [20], but the analysis on more general curves is still missing. Our results, in particular Proposition 1 and the methods developed in Appendix A, provide powerful tools in order to simplify these equations and we hope that in this way it will be possible to obtain new insights on the interplay between algebraic curves, integrable models and Riemann monodromy problem.

Appendix A

Here we prove Proposition 1 of Section 2. To this purpose, let us consider a general λ-differential $\omega\,\mathrm{d}z^\lambda$ defined on an arbitrary algebraic curve described by the two coordinates z and $y(z)$, where $y(z)$ satisfies Eq. (2.1). As it is well known, $\omega\,\mathrm{d}z^\lambda$ should be of the form

$$\omega\,\mathrm{d}z^\lambda = \frac{Q(z,y(z))}{R(z,y(z))}\,\mathrm{d}z^\lambda, \tag{A.1}$$

where Q and R are polynomials in z and $y(z)$. Moreover, using Eq. (2.1) it is always possible to make the degree of P and Q in the variable $y(z)$ equal to $n-1$ or lower. In fact, every polynomial containing powers y^m with $m \geq n$ can be transformed in a polynomial of lower degree in the obvious way:

$$y^m = -\left[P_{n-1}(z)y^{m-1} + P_{n-2}(z)y^{m-2} + \cdots + P_0(z)y^{m-n}\right]. \tag{A.2}$$

Now we show that any λ-differential $\omega\,\mathrm{d}z^\lambda$ admits the following expansion:

$$\omega\,\mathrm{d}z^\lambda = \sum_{k=0}^{n-1} g_k(z)y^k(z)\,\mathrm{d}z^\lambda, \tag{A.3}$$

where the $g_k(z)$ are rational functions of z. To this purpose, it is sufficient to rewrite the denominator $1/R(z,y(z))$ appearing in the definition of $\omega\,\mathrm{d}z^\lambda$ as a polynomial in z and y. $R(z,y)$ is of the form

$$R(z,y) = \sum_{i=1}^{n} R_{n-i}(z)y^{n-1}, \tag{A.4}$$

where the $R_{n-i}(z)$ are polynomials in the variable z only. Now we set $\tilde{y} \equiv R(z,y(z))$. $\tilde{y}$ is a multivalued function with the monodromy properties of y. By Eq. (A.2), the powers $\tilde{y}, \tilde{y}^2, \ldots, \tilde{y}^n$ remain polynomials in y and z at most of order $n-1$ in y. As a consequence, there must exist $n+1$ polynomials $\tilde{P}_i(z)$, $i = 0, \ldots, n$, such that the following Weierstrass polynomial is satisfied:

$$\tilde{P}_n(z)\tilde{y}^n + \tilde{P}_{n-1}(z)\tilde{y}^{n-1} + \cdots + \tilde{P}_1(z)\tilde{y} + \tilde{P}_0(z) = 0. \tag{A.5}$$

We notice that in some very peculiar cases, the polynomial $R(z, y)$ may not faithfully represent the local monodromy group G of y. In this case, the local monodromy group of $\tilde{y}$ is a subgroup of G and the degree of the polynomial (A.5) in $\tilde{y}$ is lower than n. We exclude also the possibility that $\tilde{P}_0 = 0$, because it is easy to see that this would imply that the Weierstrass polynomial (A.5) is reducible and one of its solutions is $\tilde{y} = 0$. But this cannot be true because, from the beginning, we suppose that the denominator of Eq. (A.1) does not identically vanish. Remembering that $R(z, y(z)) = \tilde{y}$ by definition and using Eq. (A.5), it is now easy to see that

$$\omega\,\mathrm{d}z^\lambda = -\frac{Q(z, y(z))}{\tilde{P}_0(z)}\left[\tilde{P}_n(z)R^{n-1} + \tilde{P}_{n-1}(z)R^{n-2} + \cdots + \tilde{P}_1(z)\right]\mathrm{d}z^\lambda,$$

where R has been defined in Eq. (A.4). It is then clear that the right-hand side of the above equation is a finite linear combination containing positive powers of y. The coefficients of the combination are rational functions of z. Therefore it is easy to bring $\omega\,\mathrm{d}z^\lambda$ in the desired form (A.3) using again Eq. (A.2) iteratively.

Eq. (A.3) means that an arbitrary meromorphic λ-differential can be expanded in the generalized Laurent basis whose elements are given by:

$$G_{i,k}(z, y(z))\,\mathrm{d}z^\lambda = z^i y^k(z)\,\mathrm{d}z^\lambda, \quad i = 0, 1, \ldots, \quad k = 0, \ldots, n-1. \tag{A.6}$$

To complete the proof of Proposition 1, we just need to show that the basis (2.9) is equivalent to that of Eq. (A.6). Indeed, the elements $B_{i,k}(z, y(z))\,\mathrm{d}z^\lambda$ of Eq. (2.9) are λ-differentials of the form (A.1) and can be rewritten as linear combinations of the $G_{i,k}(z, y(z))\,\mathrm{d}z^\lambda$ as in Eq. (A.3). Conversely, given an element $G_{i,k}(z, y(z))\,\mathrm{d}z^\lambda$ it is always possible to express it as a linear combination of the $B_{i,k}(z, y(z))\,\mathrm{d}z^\lambda$. As a matter of fact, the equation

$$G_{i,k}(z, y(z))\,\mathrm{d}z^\lambda = \sum_{i'\in\mathbb{Z}}\sum_{k'=0}^{n-1} a_{ii'}b_{kk'}B_{i',k'}(z, y(z))\,\mathrm{d}z^\lambda, \tag{A.7}$$

where $a_{ii'}$ and $b_{kk'}$ are constants, amounts to the following identity:

$$[F_y(z, y)]^\lambda z^i y^k = \sum_{i'\in\mathbb{Z}}\sum_{k'=0}^{n-1} a_{ii'}b_{kk'}z^{i'}y^{k'}.$$

But $[F_y(z, y)]^\lambda$ is again a polynomial in z and y which can be reduced by means of Eq. (A.2), so that there exists a finite number of constants $a_{ii'}$ and $b_{kk'}$ satisfying Eq. (A.7). This concludes the proof of Proposition 1. An analogous result can be found for the basis of the c fields (2.10).

Appendix B

The proof of the Eq. (3.13) becomes quite straigthforward once two basic lemmas are established.

Lemma B.1.

$$\prod_{i=1}^{N}(\sum_{j=1}^{M} a_{i,j}) = \sum_{r_1+\cdots+r_M=N} \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{k=1}^{M} \prod_{l_k=\alpha(k)}^{\beta(k)} a_{\sigma(l_k),k}, \tag{B.1}$$

where $a_{k,j}$ are anticommuting Grassmann variables,

$$r_j \geq 0; \quad r_0 = 0; \quad \alpha(k) = 1 + \sum_{m=1}^{k-1} r_m; \quad \beta(k) = \alpha(k) + r_k - 1. \tag{B.2}$$

The symbol $\sum_\sigma$ in (B.1) denotes the sum over all the permutations of numbers $1, \ldots, N$ such that

$$\sigma(\alpha(k)) < \cdots < \sigma(\beta(k)), \quad k = 1, \ldots, M. \tag{B.3}$$

The simplest way to prove (B.1) is to perform an induction in M.

Lemma B.2. *Consider an $N \times N$ matrix A with commuting elements $a_{i,k}$. i is the row index, while k counts the columns of A. We note that, contrary to the notation exploited in Lemma,* B. 1 *the elements $a_{i,k}$'s of the matrix A represent now non-Grassmann variables.*

Suppose we have a partition of N into M integers: $N = r_1 + r_2 + \cdots + r_M$ where $r_j \geq 0$. For each permutation σ satisfying the condition (B.3) we define matrices:

$$A_\sigma^{(k)} = \begin{pmatrix} a_{\sigma(\alpha(k)),\alpha(k)} & \cdots & a_{\sigma(\alpha(k)),\beta(k)} \\ \vdots & \ddots & \vdots \\ a_{\sigma(\beta(k)),\alpha(k)} & \cdots & a_{\sigma(\beta(k)),\beta(k)} \end{pmatrix}, \tag{B.4}$$

where $\alpha(k)$ and $\beta(k)$ are defined in (B.2). Then

$$\det A = \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{j=1}^{M} \det|A_\sigma^{(j)}|, \tag{B.5}$$

where σ is an arbitrary permutation satisfying (B.3). This is a generalization of the usual way of computing determinants (see for example [30]). The best known subcase of Eq. (B.5) is that for which $r_1 = 1$, $r_2 = N - 1$ and the remaining r_j vanish.

With the help of Lemmas B.1 and B.2 we can check Eq. (3.13). Our aim is to calculate the correlator $\langle 0| \prod_{I=1}^{N_b} b(z_I)|0\rangle$. The strategy is to decompose the fields $b(z_I)$ according to (3.9). We apply now the Lemma B.1 and obtain a sum over all possible partitions of the number N_b. According to properties of the vacuum expressed in (3.11) only one partition gives a nonzero contribution: the number of b_k fields has to be equal N_{b_k}. Once we establish this fact we arrive immediately at the product of n correlators of the type given in Eq. (3.12). With the help of Lemma B.2 we immediately complete the proof of (3.13).

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